MATH3210 - SPRING 2022 - SECTION 004

HOMEWORK 8 - *SOLUTIONS*

Problem 1. We say that a partition $\mathcal{P}$ of $[-a,a]$ is *symmetric* if $0 \in \mathcal{P}$ and whenever $x \in \mathcal{P}$, $-x \in \mathcal{P}$. Let $f : [-a,a] \to \mathbb{R}$ be an integrable function. Show that for every $\varepsilon > 0$, there exists a symmetric partition $\mathcal{P}$ such that $U(f,\mathcal{P}) - \varepsilon < \int_{-a}^{a} f\,dx < L(f,\mathcal{P}) + \varepsilon$.

Solution. Since f is integrable, there exists a partition $\mathcal{Q}$ of $[-a,a]$ such that $U(f,\mathcal{Q}) - \varepsilon < \int_{-a}^{a} f\,dx < L(f,\mathcal{Q}) + \varepsilon$. If $\mathcal{Q} = \{-a = x_0, \dots, x_N = a\}$, let $\mathcal{P} = \mathcal{Q} \cup -\mathcal{Q} \cup \{0\}$, where $-Q = \{-a = -x_N, \dots, -x_0 = a\}$. By definition, $\mathcal{P}$ is symmetric, and $L(f,\mathcal{Q}) < L(f,\mathcal{P}) < U(f,\mathcal{P}) < L(f,\mathcal{Q})$ since $\mathcal{P}$ refines $\mathcal{Q}$. Hence,

$$U(f,\mathcal{P}) - \varepsilon < \int_{-a}^{a} f\,dx < L(f,\mathcal{P}) + \varepsilon.$$

Informally, we have simply refined $\mathcal{Q}$ to force it to be symmetric. □

Problem 2. Let $f : [-a,a] \to \mathbb{R}$ be an *even* function. That is, a function such that $f(x) = f(-x)$. Show that if f is integrable, then

$$\int_{-a}^{a} f\,dx = 2\int_{0}^{a} f\,dx.$$

[*Hint*: Use the previous problem]

Solution. Let $\mathcal{P}$ be a symmetric partition of $[-a,a]$ such that $U(f,\mathcal{P})$ and $L(f,\mathcal{P})$ are within $\varepsilon/4$ of $\int_{-a}^{a} f\,dx$. and hence within $\varepsilon/2$ of each other (such a choice is possible by the previous problem). By refining as necessary, we may further assume that if $\mathcal{P}_+ = \mathcal{P} \cap [0,a]$ is the corresponding partition of $[0,a]$, then $U(f,\mathcal{P}_+)$ and $L(f,\mathcal{P}_+)$ are within $\varepsilon/8$ of $\int_0^a f\,dx$. Denote $\mathcal{P}_+ = \{0 = x_0, \dots, x_N = a\}$, so $\mathcal{P} = \{-a = -x_N, \dots, -x_1, 0, x_1, \dots, x_N\}$. Then notice that the supremum of f on $[-x_{i+1}, -x_i]$ is the same as the supremum of f over $[x_i, x_{i+1}]$, since f is symmetric, and the corresponding lengths are the same. Therefore, $U(f,\mathcal{P}) = 2U(f,\mathcal{P}_+)$, and the same formula holds for the lower sums. Combining these estimates yields.

$$\int_{-a}^{a} f\,dx - 2\int_{0}^{a} f\,dx < L(f,\mathcal{P}) + \varepsilon/4 - 2(U(f,\mathcal{P}_+) - \varepsilon/8) = L(f,\mathcal{P}) - U(f,\mathcal{P}) + \varepsilon/2 < \varepsilon$$

Similalry,

$$2\int_{0}^{a} f\,dx - \int_{-a}^{a} f\,dx < 2(L(f,\mathcal{P}_+) + \varepsilon/8) - (U(f,\mathcal{P}) - \varepsilon/4) = L(f,\mathcal{P}) - U(f,\mathcal{P}) + \varepsilon/2 < \varepsilon$$

Hence $\left|\int_{-a}^{a} f\,dx - 2\int_0^a f\,dx\right| < \varepsilon$, and since ε is arbitrary, we conclude that

$$\int_{-a}^{a} f\,dx = 2\int_{0}^{a} f\,dx.$$

□

Problem 3. Show directly that the function $f(x) = x^2$ is integrable on $[-1, 1]$, and compute $\int_{-1}^{1} f\,dx$. *Do not appeal to the theorem that every continuous function is integrable, every monotone function is integrable or the fundamental theorem of calculus.* You may use the previous problem and the following formula:

$$\sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$$

Solution. Let $\mathcal{P}_n$ denote the partition of $[0, 1]$ given by $\mathcal{P}_n = \{0, 1/n, \ldots, (n-1)/n, n\}$. Then the length of each partition interval is always $1/n = (k+1)/n - k/n$. Furthermore, the maximum value of f is always achieved at the right hand endpoint, and the minimum value is always achieved at the left hand endpoint. That is,

$$M_k = f((k+1)/n) = (k+1)^2/n^2 \qquad m_k = f(k/n) = k^2/n^2.$$

Therefore,

$$L(f, \mathcal{P}_n) = \sum_{k=0}^{n-1} \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6n^3}$$

$$U(f, \mathcal{P}_n) = \sum_{k=0}^{n-1} \frac{(k+1)^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=0}^{n-1} (k+1)^2 = \frac{n(n+1)(2n+1)}{6n^3}$$

Hence, both the upper and lower sums converge to $2/6 = 1/3$. This shows that $\int_0^1 x^2\,dx = 1/3$, so by the previous problem, $\int_{-1}^{1} x^2\,dx = 2/3$. □

Problem 4. Let f be an function on $[a, b]$ such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$.

i) Show that $\left|f(x)^2 - f(y)^2\right| \leq 2B\,|f(x) - f(y)|$ for all $x, y \in [a, b]$.
ii) Show that for any partition $\mathcal{P}$ of $[a, b]$, $U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) \leq 2B(U(f, \mathcal{P}) - L(f, \mathcal{P}))$.
iii) Show that if f is integrable on $[a, b]$, then so is f^2.

Solution.

i) $\left|f(x)^2 - f(y)^2\right| = |(f(x) - f(y))(f(x) + f(y))| \leq |f(x) - f(y)|\,(|f(x)| + |f(y)|) \leq 2B\,|f(x) - f(y)|$
ii) Fix a partition $\mathcal{P}$, and set M_i^g m_i^g to be the supremum and infemum of $g = f^2$ on $[x_i, x_{i+1}]$, respectively. Similarly, M_i^f and m_i^f denote the supremum of f on $[x_i, x_{i+1}$, respectively. Then by the previous part, $M_i^g - m_i^g \leq 2B(M_i^f - m_i^f)$. Adding these inequalities over all partition elements shows that $U(g, \mathcal{P}) - L(g, \mathcal{P}) \leq 2B(U(f, \mathcal{P}) - L(f, \mathcal{P})$.
iii) Fix $\varepsilon > 0$. Since f is integrable, there exists a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon/(2B)$. Then $U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) < 2B(U(f, \mathcal{P}) - L(f, \mathcal{P})) = \varepsilon$. Since ε was arbitrary, this shows that f^2 is integrable.

□